Selected Questions From The 1997 AIME

Click on the question to see the answers

Question 1
How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?

Question 3
Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

Question 6
Point B is in the exterior of the regular n-sided polygon $A_1A_2\ldots A_n$, and A_1A_2B is an equilateral triangle. What is the largest value of n for which A_n, A_1, and B are consecutive vertices of a regular polygon?

Question 11

Let $x = \dfrac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

Question 12
The funciton f defined by $f(x) = \dfrac{ax + b}{cx + d}$, where a, b, c, and d are nonzero real numbers, has the properties $f(19) = 19$, $f(97) = 97$, and $f(f(x)) = x$ for all values of x except $-d/c$. Find the unique number that is not in the range of f.

WRITE TO US

Correspondence about the problems and solutions for the 1997 AIME should be addressed to:

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Selected Questions From The 1998 AIME

Click on the question to see the answers

Question 1

For how many values of k is 12^{12} the least common multiple of the positive integers 6^6, 8^8, and k?

Question 3

The graph of $y^2 + 2xy + 40|x| = 400$ partitions the plane into several regions. What is the area of the bounded region?

Question 7

Let n be the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $\sum_{i=1}^{4} x_i = 98$. Find $\frac{n}{100}$.

Question 9

Two Mathematicians take a morning coffee break each day. They arrive at the cafeteria independently, at random times between 9 a.m. and 10 a.m., and stay for exactly m minutes. The probability that either one arrives while the other is in the cafeteria is 40%, and $m = a - b$ (square root c), where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

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Selected Questions From The 1999 AIME

Click on the question to see the answers

Question 1

Find the smallest prime that is the fifth term of an increasing arithmetic sequence, all four preceding terms being also prime.

Question 3

Find the sum of all positive integers n for which $n^2 - 19n + 99$ is a perfect square.

Question 5

For any positive integer x, let $S(x)$ be the sum of the digits of x, and let $T(x)$ be $|S(x+2) - S(x)|$. For example, $T(199) = |S(201) - S(199)| = |3 - 19| = 16$. How many values $T(x)$ do not exceed 1999?

Question 8

Let T be the set of ordered triples (x, y, z) of nonnegative real numbers that lie in the plane $x + y + z = 1$. Let us say that (x, y, z) *supports* (a, b, c) when exactly two of the following are true: $x \geq a, y \geq b, z \geq c$. Let S consist of those triples in T that support $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$. The area of S divided by the area of T is m / n, where m and n are relatively prime positive integers. Find $m + n$.

Question 12

The inscribed circle of triangle ABC is tangent to $\overline{AB}$ at P, and its radius is 21. Given that $AP = 23$ and $PB = 27$, find the perimeter of the triangle.

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A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person.

2007 AIME 1, Problem #2—

"List equations with variable t for Al, Bob, or Cy be the middle person, and solve for t."

Solution(052)

Let t be Al's travel time. Then $t-2$ is Bob's time, and $t-4$ is Cy's time, and $t \geq 4$. If Cy is in the middle, then $10(t-2)-8(t-4) = 8(t-4)-6t$, which has no solution. If Bob is in the middle, then $10(t-2)-8(t-4) = 6t-10(t-2)$, which has solution $t = 4/3$. But $t \geq 4$, so this is impossible. If Al is in the middle, then $6t-8(t-4) = 10(t-2)-6t$, which has solution $t = 26/3$. In this case, Al is 52 feet from the start and is 44/3 feet from both Bob and Cy. Thus the required distance is 52.

Difficulty: Easy

NCTM Standard: Algebra Standard: Represent and analyze mathematical situations and structures using algebraic symbols

Mathworld.com Classification: Number Theory > Arithmetic > General Arithmetic

The formula for converting a Fahrenheit temperature F to the corresponding Celsius temperature C is $C = \frac{5}{9}(F - 32)$. An integer Fahrenheit temperature is converted to Celsius and rounded to the nearest integer; the resulting integer Celsius temperature is converted back to Fahrenheit and rounded to the nearest integer. For how many integer Fahrenheit temperatures T with $32 \le T \le 1000$ does the original temperature equal the final temperature?

2007 AIME 1, Problem #5—

"First note that a temperature T converts back to T if and only if $T+9$ converts back to $T+9$. Explore the behavior of the first nine consecutive numbers, and note that this behavior is going to repeat itself for every nine consecutive number cycle."

Solution(539)

Note that a temperature T converts back to T if and only if $T+9$ converts back to $T+9$. Thus it is only necessary to examine nine consecutive temperatures. It is easy to show that 32 converts back to 32, 33 and 34 both convert back to 34, 35 and 36 both convert back to 36, 37 and 38 both convert back to 37, and 39 and 40 both convert back to 39. Hence out of every nine consecutive temperatures starting with 32, five are converted correctly and four are not. For $32 \le T < 32 + 9 \cdot 107 = 995$. There are $107 \cdot 5 = 535$ temperatures that are converted correctly. The remaining six temperatures $995, 996, \ldots, 1000$ behave like $32, 33, \ldots, 37$, so four of the remaining six temperatures are converted correctly. Thus there is a total of $535 + 4 = 539$ temperatures.

OR

Because one Fahrenheit degree is 5/9 of a Celsius degree, every integer Celsius temperature is the conversion of either one or two Fahrenheit temperatures (nine Fahrenheit temperatures are being converted to only five Celsius temperatures) and converts back to one of those temperatures. The Fahrenheit temperatures 32 and 1000 convert to 0 and 538, respectively, which convert back to 32 and 1000. Therefore there are 539 Fahrenheit temperatures with the required property, corresponding to the integer Celsius temperatures from 0 to 538.

Difficulty: Medium

NCTM Standard: Number and Operations Standard: Compute fluently and make reasonable estimates.

Mathworld.com Classification: Number Theory > Arithmetic > General Arithmetic

Let a sequence be defined as follows: $a_1 = 3$, $a_2 = 3$, and for $n \geq 2$,
$a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\dfrac{a_{2007}^2 + a_{2006}^2}{a_{2007}a_{2006}}$.

2007 AIME 1, Problem #14—
"The fact that the equation $a_{n+1}a_{n-1} = a_n^2 + 2007$ holds for $n \geq 2$ implies that $a_n a_{n-2} = a_{n-1}^2 + 2007$ for $n \geq 3$."

Solution(539)
The fact that the equation $a_{n+1}a_{n-1} = a_n^2 + 2007$ holds for $n \geq 2$ implies that $a_n a_{n-2} = a_{n-1}^2 + 2007$ for $n \geq 3$. Subtracting the second equation from the first one yields $a_{n+1}a_{n-1} - a_n a_{n-2} = a_n^2 - a_{n-1}^2$, or $a_{n+1}a_{n-1} + a_{n-1}^2 = a_n a_{n-2} + a_n^2$. Dividing the last equation by $a_{n-1}a_n$ and simplifying produces $\frac{a_{n+1}+a_{n-1}}{a_n} = \frac{a_n+a_{n-2}}{a_{n-1}}$. This equation shows that $\frac{a_{n+1}+a_{n-1}}{a_n}$ is constant for $n \geq 2$. Because $a_3 a_1 = a_2^2 + 2007$, $a_3 = 2016/3 = 672$. Thus $\frac{a_{n+1}+a_{n-1}}{a_n} = \frac{672+3}{3} = 225$, and $a_{n+1} = 225a_n - a_{n-1}$ for $n \geq 2$. Note that $a_3 = 672 > 3 = a_2$. Furthermore, if $a_n > a_{n-1}$, then $a_{n+1}a_{n-1} = a_n^2 + 2007$ implies that

$$a_{n+1} = \frac{a_n^2}{a_{n-1}} + \frac{2007}{a_{n-1}} = a_n\left(\frac{a_n}{a_{n-1}}\right) + \frac{2007}{a_{n-1}} > a_n + \frac{2007}{a_{n-1}} > a_n.$$

Thus by mathematical induction, $a_n > a_{n-1}$ for all $n \geq 3$. Therefore the recurrence $a_{n+1} = 225a_n - a_{n-1}$ implies that $a_{n+1} > 225a_n - a_n = 224a_n$ and therefore $a_n \geq 2007$ for $n \geq 4$. Finding a_{n+1} from $a_{n+1}a_{n-1} = a_n^2 + 2007$ and substituting into $225 = \frac{a_{n+1}+a_{n-1}}{a_n}$ shows that $\frac{a_n^2+a_{n-1}^2}{a_n a_{n-1}} = 225 - \frac{2007}{a_n a_{n-1}}$. Thus the largest integer less than or equal to the original fraction is 224.

Difficulty: Hard
NCTM Standard: Algebra Standard: Understand patterns, relations, and functions.
Mathworld.com Classification: Number Theory > Sequences

Problem

Set $\mathcal{A}$ consists of m consecutive integers whose sum is $2m$, and set $\mathcal{B}$ consists of $2m$ consecutive integers whose sum is m. The absolute value of the difference between the greatest element of $\mathcal{A}$ and the greatest element of $\mathcal{B}$ is 99. Find m.

Hint

Let the smallest elements of $\mathcal{A}$ and $\mathcal{B}$ be $(n+1)$ and $(k+1)$, respectively.

Solution (201)

Let the smallest elements of $\mathcal{A}$ and $\mathcal{B}$ be $(n+1)$ and $(k+1)$, respectively. Then

$$2m = (n+1) + (n+2) + \cdots + (n+m) = mn + \frac{1}{2} \cdot m(m+1), \quad \text{and}$$
$$m = (k+1) + (k+2) + \cdots + (k+2m) = 2km + \frac{1}{2} \cdot 2m(2m+1).$$

The second equation implies that $k + m = 0$. Substitute this into $|k + 2m - (n+m)| = 99$ to obtain $n = \pm 99$. Now simplify the first equation to obtain $2 = n + (m+1)/2$, and substitute $n = \pm 99$. This yields $m = -195$ or $m = 201$. Because $m > 0$, $m = 201$.

OR

The mean of the elements in $\mathcal{A}$ is 2, and the mean of the elements in $\mathcal{B}$ is $1/2$. Because the mean of each of these sets equals its median, and the median of $\mathcal{A}$ is an integer, m is odd. Thus $\mathcal{A} = \{2 - \frac{m-1}{2}, \ldots, 2, \ldots, 2 + \frac{m-1}{2}\}$, and $\mathcal{B} = \{-m+1, \ldots, 0, 1, \ldots, m\}$. Therefore $|2 + \frac{m-1}{2} - m| = 99$, which yields $|\frac{3-m}{2}| = 99$, so $|3 - m| = 198$. Because $m > 0$, $m = 201$.

2004 AIME, Problem #2

Problem

Alpha and Beta both took part in a two-day problem-solving competition. At the end of the second day, each had attempted questions worth a total of 500 points. Alpha scored 160 points out of 300 points attempted on the first day, and scored 140 points out of 200 points attempted on the second day. Beta, who did not attempt 300 points on the first day, had a positive integer score on each of the two days, and Beta's daily success ratio (points scored divided by points attempted) on each day was less than Alpha's on that day. Alpha's two-day success ratio was $300/500 = 3/5$. The largest possible two-day success ratio that Beta could have achieved is m/n, where m and n are relatively prime positive integers. What is $m + n$?

Hint

Let $\overline{PQ}$ be a line segment in set $\mathcal{S}$ that is not a side of the square, and let M be the midpoint of $\overline{PQ}$.

Solution (849)

Let Beta's scores be a out of b on day one and c out of d on day two, so that $0 < a/b < 8/15$, $0 < c/d < 7/10$, and $b + d = 500$. Then $(15/8)a < b$ and $(10/7)c < d$, so $(15/8)a + (10/7)c < b + d = 500$, and $21a + 16c < 5600$. Beta's two-day success ratio is greatest when $a + c$ is greatest. Let $M = a + c$ and subtract $16M$ from both sides of the last inequality to obtain $5a < 5600 - 16M$. Because $a > 0$, conclude that $5600 - 16M > 0$, and $M < 350$. When $M = 349$, $5a < 16$, so $a \le 3$. If $a = 3$, then $b \ge 6$, but then $d \le 494$ and $c = 346$ so $c/d \ge 346/494 > 7/10$. Notice that when $a = 2$ and $b = 4$, then $a/b < 8/15$ and $c/d = 347/496 < 7/10$. Thus Beta's maximum possible two-day success ratio is $349/500$, so $m + n = 849$.

OR

Let M be the total number of points scored by Beta in the two days. Notice first that $M < 350$, because 350 is 70% of 500, and Beta's success ratio is less than 70% on each day of the competition. Notice next that $M = 349$ is possible, because Beta could score 1 point out of 2 attempted on the first day, and 348 out of 498 attempted on the second day. Thus $m = 349$, $n = 500$, and $m + n = 849$.

Note that Beta's two-day success ratio can be greater than Alpha's while Beta's success ratio is less on each day. This is an example of Simpson's Paradox.

2004 AIME, Problem #5

Problem

A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Given that the probability that the circle will not touch diagonal $\overline{AC}$ is m/n, where m and n are relatively prime positive integers, find $m+n$.

Hint

If the circle is to lie completely within the rectangle, then where must the center of the circle lie?

Solution (817)

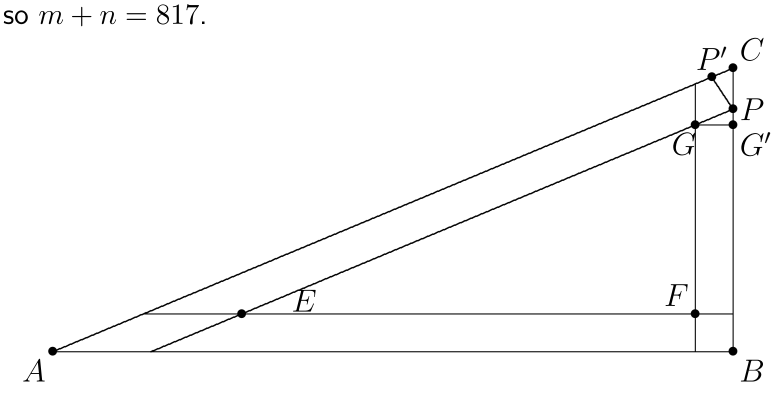
In order for the circle to lie completely within the rectangle, the center of the circle must lie in a rectangle that is $(15-2)$ by $(36-2)$ or 13 by 34. The requested probability is equal to the probability that the distance from the circle's center to the diagonal $\overline{AC}$ is greater than 1, which equals the probability that the distance from a randomly selected point in the 13-by-34 rectangle to each of the sides of $\triangle ABC$ and $\triangle CDA$ is greater than 1. Let $AB = 36$ and $BC = 15$. Draw the three line segments that are one unit respectively from each of the sides of $\triangle ABC$ and whose endpoints are on the sides. Let E, F, and G be the three points of intersection nearest to A, B, and C, respectively, of the three line segments. Let P be the intersection of $\overrightarrow{EG}$ and $\overline{BC}$, and let G' and P' be the projections of G and P on $\overline{BC}$ and $\overline{AC}$, respectively. Then $FG = BC - CP - PG' - 1$. Notice that $\triangle PP'C \sim \triangle ABC$ and $PP' = 1$, so $CP = AC/AB$. Similarly, $\triangle GG'P \sim \triangle ABC$ and $GG' = 1$, so $PG' = CB/AB$. Thus

$$FG = BC - \frac{AC}{AB} - \frac{CB}{AB} - 1.$$

Apply the Pythagorean Theorem to $\triangle ABC$ to obtain $AC = 39$. Substitute these lengths to find that $FG = 25/2$. Notice that $\triangle EFG \sim \triangle ABC$, and their similarity ratio is $(25/2)/15 = 5/6$, so $[EFG] = (25/36)[ABC]$. The requested probability is therefore

$$\frac{2 \cdot \frac{25}{36} \cdot \frac{1}{2} \cdot 15 \cdot 36}{13 \cdot 34} = \frac{375}{442},$$

so $m+n = 817$.



AMERICAN MATHEMATICS COMPETITIONS

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

ANSWER FORM

IMPORTANT DIRECTIONS FOR MARKING THIS FORM:

1. Students who are in the top 5% of all AMC 12 participants or score at least 100 points on the AMC 12, and those who are in the top 1% of the AMC 10 participants are invited to participate in the AIME.

2. Provide the information requested in Section I.

3. Print the requested information in the row of boxes in Section II below.

4. For all coded information, including your answers, USE A NO. 2 PENCIL ONLY, to completely blacken the circle. A CIRCLE IS COMPLETELY BLACKENED ONLY IF THE LETTER/ NUMBER IS NOT VISIBLE.
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5. Check the blackened circles for accuracy and erase errors completely.

6. Answers to the exam are to be marked on Side 2 of this form.

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CITY | STATE OR PROVINCE | ZIP OR POST CODE

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IF ELIGIBLE I WILL PARTICIPATE IN THE USAMO. ○ YES ○ NO

NAME OF SCHOOL

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Which previous exam taken:
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○ AMC 12

GENDER
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LAST NAME (Print) | FIRST NAME (Print) | MI

SCORE | GRADE | AGE

GRADE: ≤ 4, 5, 6, 7, 8, 9, 10, 11, 12, 13

AGE: ≤ 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

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AMERICAN MATHEMATICS COMPETITIONS
AIME Answer Form

DIRECTIONS:

1. CALCULATORS ARE NOT ALLOWED.
2. USE A NO. 2 PENCIL ONLY to mark your answers.
3. For your convenience in mark sensing your answers, write your answer to each problem in the spaces below the problem number using 3 digits in the columns reserved.
4. In each of these columns blacken the one circle which is labeled with the digit you have written at the top.
5. A single digit answer such as 7, should be written and coded as 007, and a two digit answer such as 43, should be written and coded as 043.
6. Compare the blackened circles with the boxes for accuracy. Any discrepancy between the answer to the problem as written and marked will be grounds for ruling the answer incorrect.
7. Erase any errors completely.
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9. After completing the exam, sign your name below to certify that these answers represent your own work.

Student's Signature ______________________________

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Blacken the circle above if you give the CAMC permission to release this information to these organizations. (Your score will not be affected if you do not blacken the circle.)

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Your results will be graded by computer scanning. **Only the blackened circles will be graded.**

PROBLEM 1	PROBLEM 2	PROBLEM 3	PROBLEM 4	PROBLEM 5	PROBLEM 6	PROBLEM 7	PROBLEM 8
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PROBLEM 9	PROBLEM 10	PROBLEM 11	PROBLEM 12	PROBLEM 13	PROBLEM 14	PROBLEM 15
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